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Regular Figures in n-*dimensional Space.*

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A pencil of lines, diverging from a common vertex in n-dimensional space, forms the edges of an n-fold (short for n-dimensional) angle. There must be at least n of them, for otherwise they would lie in a space of less than n dimensions. If there be just n of them, combined two and two they form 2-fold face boundaries; three and three, they form 3-fold trihedral boundaries, and so on. So that the simplest n-fold angle is bounded by n edges, $\frac{n(n-1)}{2}$ faces, $\frac{n(n-1)(n-2)}{1.2.3}$ 3-folds, in fact, by $\frac{n!}{k!(n-k)!}$ k-folds. Let such an angle be called *elementary*. Fig. 1 represents the symmetrical arrangement, in three-dimensional perspective, of the four trihedral boundaries of an elementary 4-fold angle. When put into space of four dimensions, the faces of the tetrahedra, which lie adjacent to the common vertex, are to be brought into coincidence two and two; the edges will then fall together three and three.

A regular angle is defined as one all of whose boundaries of any given number of dimensions are the same in form and magnitude. The number of regular n-fold angles that can be formed out of sets of regular $(n-1)$-fold angles is limited by the number of those symmetrical arrangements of the $(n-1)$-fold angles about a vertex in an $(n-1)$-fold space, which involve the symmetrical and equal distribution of all of their boundaries. To give concreteness to the idea, let equal distances be measured from the vertex of a regular n-fold angle on its edge boundaries. These equal lines must terminate in the summits of a regular $(n-1)$-fold figure; from which it follows that the different kinds of regular n-fold angles are just equal in number to the regular $(n-1)$-fold figures or, what is the same thing, to the regular distributions of points on the $(n-1)$-fold sphere. Hence there suggests itself an obvious criterion for the possibility of any regular n-fold angle, viz.:

(A.) *The number of its edge boundaries shall be equal to the number of summits of some regular* $(n-1)$*-fold figure.*

Conformably with a notation hereafter employed in connection with the regular figures (see p. 8), this criterion may be expressed analytically in the form
$$(n)_1^0 = (n-1)_0^{n-1}, \quad \text{or} \quad (n)_1^0 = (N-1)_0,$$
where $(N-1)_0 = (n-1)_0^{n-1}$ is the number of summits of some regular $(n-1)$-fold figure, and is to have all possible values under the conditions named, and $(n)_1^0$ is the number of edges which are to enter into the formation of the regular n-fold angle; $(n)_1^0$ can have only such values as are equal to the different possible values of $(n-1)_0^{n-1}$. The criterion is at least sufficient, and in this discussion I shall assume it to be necessary, reserving the question for future consideration, if such be required.

In particular, the number of regular 4-fold angles that can be made up of regular polyhedral angles is five. Out of trihedral angles three regular 4-fold angles can be formed, corresponding to the three regular polyhedra which have triangular boundaries. Similarly there is one regular distribution for tetrahedral and one for pentahedral angles. The former consists of a set of six angles corresponding to the faces of the cube, and the latter of twelve angles corresponding to the faces of the dodekahedron.

Three of the five regular 4-fold angles are shown in Plates I., II., laid out in three-dimensional perspective. Figs. 1 and 3 belong to what may be called the tetrahedroidal, Fig. 5 to the oktahedroidal, and Fig. 7 to the hexahedroidal 4-fold angles. The other two, not represented in the plates, are dodekahedroidal and ikosahedroïdal.

A *complete* n-fold figure is defined as one which is limited on every hand by complete $(n-1)$-fold figures which are themselves limited by complete $(n-2)$-fold figures, and so on. In other words, a complete figure has no gaps in it. It will be convenient to designate as an n-fold *polyhedroid* the n-dimensional figure which is bounded by $(n-1)$-fold flat (not curved) figures; and in particular to call the figure which has m $(n-1)$-fold boundaries an n-fold (m)-hedroid; thus, an oktahedron is a 3-fold (8)-hedroid, or oktahedroid.

An n-fold *elementary* pyramid is the simplest n-fold figure. It has $n+1$ summits, $\frac{(n+1)\,n}{2}$ edges, $\frac{(n+1)\,n\,(n-1)}{1\,.\,2\,.\,3}$ faces, and in general it has $\frac{(n+1)\,!}{k\,!\,(n-k+1)\,!}$ $(k-1)$-dimensional boundaries. Thus the successive terms of the left-hand member of
$$(1-1)^{n+1} = 0, \tag{1}$$
beginning with the second and ending with the $(n+1)^{\text{th}}$, represent the numbers

of boundaries of the figure, the second term standing for summits, the third for edges, the fourth for two-dimensional boundaries, and so on. This is, in fact, a generalized form, for this special class of n-fold figures, of Euler's equation for polyhedra. The summits of a *regular* elementary pyramid are all equidistant from each other, and may be said to have *absolute* symmetry of position with reference to each other. The elementary pyramids are self-reciprocal in the sense in which a tetrahedron is said to be self-reciprocal; that is to say, if the centres of the $(n-1)$-fold boundaries be taken for the summits of a new figure, then the old summits will be replaced by an equal number of $(n-1)$-fold boundaries, and in general the same reciprocal relation will exist between any two sets of boundaries which are equidistant from the two ends of the series; viz. a set of k-fold and a set of $(n-k-1)$-fold boundaries.

In particular, the 4-fold pentahedroid has 5 summits, 10 edges, 10 triangular and 5 tetrahedral boundaries. To construct this figure select any one summit of each of four tetrahedra and unite them. Bring the faces, which lie adjacent to each other, two and two into coincidence. There will remain four faces still free; take a fifth tetrahedron, and join each one of its faces to one of these four remaining ones. The resulting figure will be the complete 4-fold pentahedroid.

It has been seen that Equation (1) is a particular case of the generalized Eulerian polyhedral formula. Before proceeding with the discussion of the other regular figures, I propose to give a demonstration of the generalized formula for n-fold figures in general. Let N_k represent the number of k-fold boundaries to an n-fold figure; that is, say it has N_0 summits, N_1 edges, N_2 faces, etc. Conformably with this notation, Equation (1) will assume the form

$$(1-1)^{n+1} = 1-(n+1)+\frac{(n+1)\,n}{1.2} - \cdots \pm (n+1) \mp 1$$

$$= 1 - N_0 + N_1 - \cdots \pm N_{n-1} \mp N_n = 1 - \sum_{k=0} (-)^k N_k = 0, \quad (\alpha)$$

wherein N_n is to be counted as 1 when the figure is simple, as here, i. e. not an assemblage of two or more n-fold figures. It is to be shown that this equation is true for any n-fold figure whatever, simple or complex;—by a *complex* figure I mean one made up of two or more simple figures joined together. It is important to observe that, when the figure is complex, it may either be treated as such, in which case N_n is the number of figures in the group, or it may be regarded as an $(n+1)$-fold simple figure, in which case another unit is to be added to N_n, viz. some selected n-fold outline of the figure, not counted before, is to be counted as 1 in N_n. Also a term N_{n+1}, counting as 1 for the $(n+1)$-fold figure, must be added to the formula. The figure may be actually converted into a simple $(n+1)$-fold one by properly distorting it in the $(n+1)$-dimensional space. The formula thus remains the same in form for the $(n+1)$-fold simple figure as for the n-fold complex one, the upper limit of k for the former case being $n+1$, for the latter n. The n-fold complex figure is, in fact, simply the projection of the $(n+1)$-fold one into one of its n-fold boundaries. On this principle is based the following very simple proof that the function (α) must vanish for any n-fold simple figure, or what is the same thing, for any $(n-1)$-fold complex one.

For brevity, write the function (α), corresponding to a simple n-fold figure of arbitrary construction, in the form

$$\Phi(n) = 1 - \sum_{k=0} (-)^k N_k. \tag{α}$$

The object of the proof is to show that $\Phi(n) = 0$. For the purposes of the proof, the boundaries of the figure may be supposed to be elastic, so as to admit of being distorted without incurring any change in their numerical relations. First, project the figure as a whole into one of its $(n-1)$-fold boundaries; the term $N_n = \pm 1$ disappears, together with a unit, of opposite sign from N_n, out of the term N_{n-1}; these units cancel each other, and the function becomes $\Phi(n-1)$. The figure is now $(n-1)$-dimensional and complex, and N_{n-1} is the number of its $(n-1)$-fold constituents. Taking any one of these constituents, project it into one of its own $(n-2)$-fold boundaries, distorting the other boundaries of the figure sufficiently to allow of this being done; taking care, at the same time, that no two boundaries shall fall into coincidence in the projected figure, and so one of them disappear. In this process the $(n-1)$-fold figure is obliterated and also an $(n-2)$-fold one; viz., the former has collapsed and no longer exists, while the latter is replaced by the other $(n-2)$-fold figures which have been projected into it. Thus, a unit is extracted from each of the terms N_{n-1}, N_{n-2}, and the two units cancel each other in consequence of N_{n-1} and N_{n-2} having opposite signs. This operation, repeated upon all save one of the other constituent $(n-1)$-fold parts of the complex $(n-1)$-fold figure, will reduce the figure to a simple $(n-1)$-fold one, for which N_{n-1} is equal to 1. Observe that at each step of this reduction *one* unit and *only one* is extracted from each of the terms N_{n-1}, N_{n-2}. The projection can always be made so as to insure this result.*

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* It is of no consequence if some of the boundaries intersect each other in the projected figure. But even this may be avoided by a proper order in making the projections and by a proper distortion of the figure. Thus, if two $(n-1)$-fold boundaries happen to lie in the same $(n-1)$-dimensional flat space, such a crossing of boundaries will take place; but the two $(n-1)$-fold boundaries may, by a slight distortion of the figure, be made to lie in different $(n-1)$-dimensional spaces.

This simple $(n-1)$-fold figure can now be projected as a whole into one of its $(n-2)$-fold boundaries, thus becoming an $(n-2)$-fold complex figure; N_{n-1} will disappear entirely from the formula, which thus becomes $\Phi(n-2)$. By a repetition of the foregoing process, the number of dimensions of space to which the figure belongs is diminished by successive units, simultaneously with which the upper limit of k in the Φ function is also diminished by successive units. The outcome of this is that in the limit, on arriving at zero-dimensional space, the function reduces to the simple identity $\Phi(0)=1-1=0$. Now since the quantities which were extracted from the terms of $\Phi(n)$ have cancelled each other in pairs, and the function is zero in the limit, therefore, in general, $\Phi(n)=0$, which was to be proved.

I pass to the further consideration of the regular figures, and first of those belonging to four-dimensional space. The 4-fold pentahedroid has already been described. The next simplest is the 4-fold orthogonal figure; it is an oktahedroid. It may be generated by giving the 3-fold cube a motion of translation in the fourth dimension in a direction perpendicular to the three-dimensional space in which it is situated. Each summit generates an edge, each edge a square, each square a cube. To the numbers of boundaries thus generated must be added twice the numbers of summits, edges, squares, and cubes of the original figure (viz. each summit, edge, square, and cube must be counted once for its initial and once for its final position). Using D_0, D_1, D_2, D_3, D_4 in place of the N_0, N_1, ... to represent in particular the numbers of the boundaries of 4-fold figures, we have for the case in hand

$$\begin{aligned}
D_0 &= \phantom{12+{}} 8+8=16=2^4\\
D_1 &= 12+8+12=32=4\,.\,2^3\\
D_2 &= 6+12+6=24=\frac{4\,.\,3}{2}\,2^2\\
D_3 &= 1+6+1=8=\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\,2\\
D_4 &= =1.
\end{aligned}$$

The equation (α) for this case becomes

$$1-D_0+D_1-D_2+D_3-D_4=1-(2-1)^4=0.$$

By an extension of the above process of construction to higher dimensional figures it is easy to show that in general the boundaries of the n-fold orthogonal figure give the relation

$$1-N_0+N_1-N_2+N_3-\cdots\mp N_n=1-(2-1)^n, \qquad (2)$$

the upper sign corresponding to n even, the lower to n odd.

The 4-fold hexadekahedroid and its analogues in n-fold space are the reciprocals of the corresponding orthogonal figures, and the Φ function for these is (2) written backwards, i. e.

$$1 - N_0 + N_1 - N_2 + N_3 - \cdots\cdot \mp N_n = (1-2)^n \mp 1 = 0. \qquad (3)$$

But it will be interesting to see how the 4-fold figure may be completely determined by purely geometrical considerations. Take the eight middle points of the cube boundaries of the 4-fold octahedroid as summits; $D_0 = 8 = 4\,.\,2$. These points are the eight extremities of the four mutually perpendicular diameters of the 4-fold sphere. The edges of the figure are found by joining each summit with each of the other summits except its antipode, i. e. with six adjacent ones; so that the number of edges diverging from a vertex is six. Hence $D_1 = \frac{6\,D_0}{2} = \frac{4\,.\,3}{1\,.\,2}\,2^2$. Consider the four points, equidistant from each other, which lie at the extremities of four mutually perpendicular radii of the 4-fold sphere. They have absolute symmetry of position in three-dimensional space, and so are the summits of a regular tetrahedron. Any fifth point of the series is an antipode of one of the four already selected, and such a group of five points cannot lie in a space of three dimensions, so that four of the eight points only can be taken as constituting the summits of the three-dimensional boundaries; hence these boundaries are tetrahedra. Their number is found by enumerating the groups of four out of the eight points, excluding groups which contain antipodes; so that $D_3 = 1 + 4 + \frac{4\,.\,3}{2} + \frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3} + 1 = 16 = 2^4$; whence, also, $D_2 = \frac{4\,D_3}{2} = 32 = \frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\,2^3$. Therefore D_0, D_1, D_2, D_3, D_4 do satisfy the relation (3), as was predicted.

It appears, then, that every space has at least three regular figures. I shall hereafter refer to these as the binomial tri-group in n-dimensional space. The three figures in question are, in the notation we have adopted, the n-fold $(n+1)$-hedroid, the n-fold $(2\,n)$-hedroid, and the n-fold (2^n)-hedroid.

Figs. 1, 3, 5 represent respectively the summits, one in each figure, of the 4-fold pentahedroid, oktahedroid, and hexadekahedroid, with the 3-fold boundaries of the summit spread out symmetrically in three-dimensional space. Figs. 2, 4, 6 represent the complete projections of these three regular figures upon a plane. The tetrahedral boundaries of the pentahedroid (Fig. 2) are $a\,b\,c\,d$, $b\,c\,d\,e$, $c\,d\,e\,a$, $d\,e\,a\,b$, $a\,b\,c\,e$. The eight cube boundaries of the oktahedroid (Fig. 4) are $a\,b\,c\,d\,e\,f\,g\,h$, $l\,m\,n\,p\,q\,r\,s\,t$, $a\,b\,c\,d\,l\,m\,n\,p$, $e\,f\,g\,h\,q\,r\,s\,t$, $a\,c\,l\,n\,e\,g\,q\,s$, $b\,d\,m\,p\,f\,h\,r\,t$, $n\,c\,d\,p\,g\,h\,s\,t$, $a\,b\,l\,m\,e\,f\,q\,r$. The sixteen tetrahedral boundaries of the hexadekahedroid (Fig. 6) are $a\,b\,c\,d$, $a'b'c'd'$, $a\,b'c'd'$, $a'b\,c\,d$, $a\,b'c\,d$, $a'b\,c'd'$,

$abc'd$, $a'b'cd'$, $abcd'$, $a'b'c'd$, $abc'd'$, $a'b'cd$, $a'bc'd$, $ab'cd'$, $a'bcd'$, $ab'c'd$. The accented letters are the antipodes of the unaccented ones.

Now in building up all possible 4-fold regular figures we have five regular 3-fold figures to deal with. Let us take successively these five figures, and using each as a framework upon which to arrange symmetrically the 3-fold boundaries to one of the summits of the 4-fold figures, consider all the possible cases. (The framework above referred to I shall call the *frame* figure, and the $(n-1)$-fold figure, out of a group of which the n-fold one is to be constructed, the *generating* figure.) The arrangements, with the 3-fold figures, which give regular distributions amongst the summits of the frame figure, i. e. those which satisfy the criterion (A) of page 2, for a regular angle, are —

(*i.*) 4 tetrahedra upon a tetrahedral frame.
(*ii.*) 4 hexahedra " " "
(*iii.*) 4 dodekahedra " " "
(*iv.*) 8 tetrahedra upon an octahedral frame.
(*v.*) 8 hexahedra " " "
(*vi.*) 8 dodekahedra " " "
(*vii.*) 20 tetrahedra " ikosahedral "
(*viii.*) 20 hexahedra " " "
(*ix.*) 20 dodekahedra " " "
(*x.*) 6 octahedra upon a hexahedral "
(*xi.*) 12 ikosahedra " dodekahedral "

These eleven cases naturally fall into reciprocal pairs wherein the generating figure of the one case is the reciprocal of the frame figure of the other, and into self-reciprocal single categories in which the generating and frame figures are the reciprocals of each other. The reciprocal pairs are (*ii*) (*iv*), (*iii*) (*vii*), (*vi*) (*viii*), and the self-reciprocal categories are (*i*), (*v*), (*ix*), (*x*), (*xi*). The resulting figures will group themselves into three classes, — real, imaginary, and infinite. The imaginary figures are those which cannot be built up in consequence of their enclosing too much space; the infinite ones are those which completely fill up — saturate, so to speak — infinite three-dimensional space.

For the rejection of those cases which give rise to imaginary figures and the complete determination of the real figures, the following general criteria, applicable to an n-fold regular figure, will be useful, and are obviously necessary: —

(B.) *The number of* $(n-1)$*-fold boundaries which border upon any summit of a regular* n*-fold figure must be less than the number of those* $(n-1)$*-fold boundaries which can be joined together — with their* $(n-2)$*-fold boundaries coincident two and two — about a point in* $(n-1)$*-dimensional space.*

(C.) *In a regular* n-*fold figure, every boundary of a given number of dimensions must have lying adjacent to it the same number of equal boundaries of any other number of dimensions.*

Analytically expressed, the criterion (C) is equivalent to

$$(n)_r^s N_s = (n)_s^r N_r, \qquad (4)$$

where N_s, N_r represent the number of s- and r-fold boundaries to the regular figure, and $(n)_r^s$ = the number of r-fold boundaries to an s-fold boundary, $(n)_s^r$ = the number of s-fold boundaries to an r-fold one. Here n is the number of dimensions of the space to which the figure belongs, and r, s may have any values from 0 to n inclusive. It is obvious that $(n)_k^n = N_k$.

For the application of the criterion (B) to the 4-fold figures, the following table of lengths of edges of the five regular solids, each inscribed in a sphere whose radius is unity, will be useful : —

Tetrahedron, $l = 1.632994$,
Hexahedron, $l = 1.154700$,
Oktahedron, $l = 1.414214$,
Dodekahedron, $l = .713644$,
Ikosahedron, $l = 1.051462$.

Case (v) gives the only infinite figure in the above scheme. Its construction is obvious.

Cases (i), (ii), (iv) give real figures, viz. the pentahedroid, the oktahedroid, and the hexadekahedroid.

Cases (vi), (ix). Let the three summits of a dodekahedron, which are nearest to and equidistant from another of its summits S, be placed in coincidence with three mutually adjacent summits of a regular oktahedron inscribed in the unit sphere, and let the summit S be directed inward towards the centre of the sphere. The length of an edge of the oktahedron being 1.414214, the calculated length of edge of the dodekahedron is .874048, or less than the radius of the sphere. Hence the summit S will fall short of the centre of the sphere; wherefore eight trihedral angles of regular dodekahedra more than fill up the three-dimensional space about a point. Case (vi), therefore, gives rise to an imaginary figure. Case (ix) likewise gives an imaginary figure, since l is less for the ikosahedron than for the oktahedron.

Case ($viii$). This, being the reciprocal of Case (vi), must give an imaginary figure. It is otherwise seen to be imaginary from the fact that the length of edge of the frame figure (the ikosahedron) being 1.051462, that of the hexahedron is $1.051462 \div \sqrt{2} = .743496$, or less than the radius of the sphere.

Case (xi). Five summits of an ikosahedron adjacent to another summit S are to be placed in coincidence with the five summits of a pentagonal face of the frame dodekahedron, with S inside the sphere. The length of edge of the dodekahedron being .713644, that of the ikosahedron is the same, or less than the radius of the sphere. Hence the figure is imaginary.

Case (x). Place the four summits of an oktahedron (Fig. 6), which lie at the corners of a square section, in coincidence with the four corners of one of the faces of the inscribed cube. The edge of the cube is 1.1547, greater than the radius of the sphere. Hence this case gives rise to a real figure.

Case (iii). The length of an edge of the frame figure (the tetrahedron) being 1.632994, the calculated length of an edge of the dodekahedron is 1.009261, or greater than the radius of the sphere. The figure is real.

Case (vii). The length of edge of the frame figure (the ikosahedron) is 1.051462, and that of the tetrahedron is the same, or greater than the radius of the sphere. The figure is real.

There are thus three 4-fold figures yet to be determined, the self-reciprocal figure of Case (x) and the reciprocal group of Cases (iii), (vii).

That the figure of Case (x) is self-reciprocal is shown by the relations

$$(d)^3_0\, D_3 = (d)^0_3\, D_0\,,\ (d)^2_1\, D_2 = (d)^1_2\, D_1\,,$$

special forms of the equation of Criterion (C), wherein for the case in hand

$$(d)^2_1 = (d)^1_2 = 3\,,\ (d)^3_0 = (d)^0_3 = 6.$$

The figure is a 4-fold ikosatetrahedroid. A summit out of each of six oktahedra joined together make up one summit of the new figure. Fig. 7 shows such a summit with six oktahedral boundaries arranged about it symmetrically in three-dimensional space. Conceive Fig. 7 to be transported into four-dimensional space and the interstices between the adjacent triangular faces to be closed up by joining those faces two and two; the figure assumes a form whose projection is represented in Fig. 8 with dotted lines omitted. Adjust to this figure twelve other oktahedra in a symmetrical manner; three of these oktahedra are represented by the dotted lines of Fig. 8. Again, close up the interstices between the adjacent faces; the outline of the figure assumes a form whose projection is represented in Fig. 9. Now, conceive this figure to be turned inside out. There will be left in the middle of the figure a vacant space of exactly the form of Fig. 8 with the dotted lines omitted; such a group of six oktahedra is therefore required to complete the 4-fold figure. By counting up all the constituent oktahedral summits and other boundaries the reader may satisfy himself that the summits of the 4-fold figure are filled to saturation, and that the figure is in other respects complete and regular.

The number of oktahedral boundaries is $D_3 = 6 + 12 + 6 = 24$; of summits, $D_0 = \frac{6 \,.\, 24}{6} = 24$; of triangular faces, $D_2 = \frac{8 \,.\, 24}{2} = 96$; of edges, $D_1 = \frac{8 \,.\, 24}{2} = 96$. Thus the equation $\Phi(n)$ for this case is

$$1 - D_0 + D_1 - D_2 + D_3 - D_4 = 1 - 24 + 96 - 96 + 24 - 1 = 0. \qquad (5)$$

This figure bears peculiar relations to the oktahedroid and the hexadekahedroid. It is easily shown that the edge of the oktahedroid is equal to the radius of the circumscribed 4-fold sphere; in fact, $4\,l^2 = (2\,r)^2$, or $l = r$, where r is the radius of the sphere and l is the length of edge of the oktahedroid. Now, the semidiagonal of the cube whose edge is r, is $r\,\frac{1}{2}\sqrt{3}$; therefore, the distance from the centre of the 4-fold sphere to the centre of one of the cube boundaries of the inscribed regular oktahedroid is $\sqrt{r^2 - r^2(\frac{1}{2}\sqrt{3})^2} = \frac{1}{2}\,r$. Hence the point of intersection, P, of the sphere with the radius drawn through the centre of one of these cubes is at a distance r from each of the summits of the cube; in other words, the eight lines joining this point with the summits of the cube are in length equal to an edge of the cube, therefore also to the radius of the sphere. These lines, in fact, are edges of a regular ikosatetrahedroid. There are $8 \times 8 = 64$ of the lines joining the points P with the summits of the oktahedroid, and these together with the 32 edges of the oktahedroid constitute the 96 edges of the ikosatetrahedroid. Moreover, the summits of the latter figure are the 16 summits of the oktahedroid plus the 8 points P. Again, each of the 32 edges of the oktahedroid has 3 cubes bounding it; hence, there are 3 triangular faces of the ikosatetrahedroid corresponding to each of these edges, or in all $3 \times 32 = 96$. Finally, the 24 2-fold faces of the oktahedroid are 24 square plane sections of the oktahedral boundaries of the ikosatetrahedroid. Eight of the summits of the ikosatetrahedroid constitute the summits of a regular hexadekahedroid; but the edges and 2-fold boundaries of the latter figure are quite distinct from those of the former.

A projection of the ikosatetrahedroid may be constructed by drawing all the diagonal lines of the 8 cubes of Fig. 6, and regarding the half-diagonals as edges of the new figure and the middle points of diagonals as summits.

Having determined the figure of either Case (*iii*) or (*vii*), that of the other will be determined as its reciprocal. The process of building up the figure of Case (*vii*) is graphically illustrated in Figs. 10–17. Twenty tetrahedral summits are required to make up a summit of the 4-fold figure. By joining together a single group of 20 tetrahedra and bringing the adjacent faces two and two into coincidence, a figure is produced whose projection in three-dimensional space has the form of Fig. 10.* In Fig. 11, 20 more tetrahedra have been added; the addition of 20 more to Fig. 11 gives Fig. 12; the addition of $5 \times 12 = 60$ to Fig. 12 gives Fig. 13; the addition of $2 \times 30 = 60$ to Fig. 13 gives Fig. 14; the addition of $5 \times 12 = 60$ to Fig. 14 gives Fig. 15. The manner of distributing the tetrahedra as they are added is obvious. This last figure contains 20 indentations into which trihedral, and 12 into which pentahedral angles must be inserted. Form 20 groups of 2 tetrahedra each, as represented in Fig. 16, having trihedral angles at R, R'; and form 12 groups of 15 tetrahedra each, as represented in Fig. 17, having pentahedral angles at S, S'. These $20 + 12 = 32$ groups, together with two of the groups of Fig. 15, will fit into each other and complete the regular 4-fold figure. That the summits of this figure are all saturated can be verified by actual count. Thus the number of tetrahedral boundaries to the regular figure is $2\,(20 + 20 + 30 + 60 + 60 + 60) + 2 \times 20 + 5 \times 12 = 600$. Applying the criterion (C) in order to determine the other boundaries, we have

$$(d)^0_3 = 20, \quad (d)^3_0 = 4, \quad (d)^1_3 = 5, \quad (d)^3_1 = 6, \quad (d)^2_3 = 2, \quad (d)^3_2 = 4;$$

whence, in consequence of the relation $(d)^s_r\, D_s = (d)^r_s\, D_r$,

$$\begin{aligned} 20\,D_0 &= 4 \times 600, & D_0 &= 120; \\ 5\,D_1 &= 6 \times 600, & D_1 &= 720; \\ 2\,D_2 &= 4 \times 600, & D_2 &= 1200; \end{aligned}$$

so that $\Phi\,(n)$, for this case, is

$$1 - D_0 + D_1 - D_2 + D_3 - D_4 = 1 - 120 + 720 - 1200 + 600 - 1 = 0. \qquad (6)$$

In the nomenclature we have adopted the figure is a 4-fold (600)-hedroid, or hexakosioihedroid.

The reciprocal of the figure last described is evidently a 4-fold (120)-hedroid, or hekatonikosihedroid. This figure may be constructed as follows: Each summit must have 4 constituent dodekahedral summits. Take a single dodekahedron as a foundation upon which to build. Surround it with 12 other dodekahedra, as represented in Fig. 18. The successive additions to this figure of 20, 12, and 30 dodekahedra produce respectively the figures 19, 20, and 21. Now 19 and 20 are the negatives of each other in the photographic sense, and will fit together so as to complete the regular figure. Thus the number of dodekahedral boundaries to the figure is $2\,(1 + 12 + 20 + 12) + 30 = 120$. Applying the criterion (C) we have

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* The constituent tetrahedra represented in the projected figures are not regular in so far as they have not equal regular faces. But that is evidently of no consequence, so long as the interstices are closed in the proper way and the right faces are left free to admit of further additions. In other respects the projected figures may be distorted *ad libitum.* Figs. 11, 13 are the two regular star-faced dodekahedra of Poinsot.

$$(d)_3^0 = 4, \quad (d)_0^3 = 20, \quad (d)_3^1 = 3, \quad (d)_1^3 = 30, \quad (d)_3^2 = 2, \quad (d)_2^3 = 12\,;$$

$$\begin{aligned} 4\,D_0 &= 20 \times 120, & D_0 &= 600\,; \\ 3\,D_1 &= 30 \times 120, & D_1 &= 1200\,; \\ 2\,D_2 &= 12 \times 120, & D_2 &= 720\,; \end{aligned}$$

$$1 - D_0 + D_1 - D_2 + D_3 - D_4 = 1 - 600 + 1200 - 720 + 120 - 1 = 0, \qquad (7)$$

which last is (6) written backwards, as it should be.

Passing to the discussion of the regular 5-fold figures, the several arrangements to be considered are

(*i.*)	5	(5)-hedroids	upon	a	(5)-hedroidal	frame.
(*ii.*)	5	(8)-hedroids	"	"	"	"
(*iii.*)	5	(120)-hedroids	"	"	"	"
(*iv.*)	16	(5)-hedroids	"	"	(16)-hedroidal	"
(*v.*)	16	(8)-hedroids	"	"	"	"
(*vi.*)	16	(120)-hedroids	"	"	"	"
(*vii.*)	8	(24)-hedroids	"	"	(8)-hedroidal	"
(*viii.*)	24	(16)-hedroids	"	"	(24)-hedroidal	"
(*ix.*)	600	(5)-hedroids	"	"	(600)-hedroidal	"
(*x.*)	600	(8)-hedroids	"	"	"	"
(*xi.*)	600	(120)-hedroids	"	"	"	"
(*xii.*)	120	(600)-hedroids	"	"	(120)-hedroidal	"

The reciprocal pairs in this scheme are (*ii*) (*iv*), (*vii*) (*viii*), (*iii*) (*ix*), and the cases giving self-reciprocal figures are (*i*), (*v*), (*xi*), (*xii*).

For present purposes the following approximate lengths of edges of the 4-fold regular figures inscribed in the unit 4-fold sphere are sufficiently accurate:

4-fold (5)-hedroid, $l = 1.58113$,
" (8)-hedroid, $l = 1.00000$,
" (16)-hedroid, $l = 1.41421$,
" (24)-hedroid, $l = 1.00000$,
" (120)-hedroid, $l < 1$,
" (600)-hedroid, $l < 1$.*

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* In general, the length of edge of the n-fold $(n + 1)$-hedroid, $(2\,n)$-hedroid and (2^n)-hedroid are respectively

$$r_n \sqrt{\frac{2\,(n+1)}{n}}, \quad r_n \frac{2}{\sqrt{n}}, \quad r_n \sqrt{2},$$

where r_n is the radius of the circumscribed n-fold sphere. The first expression is a consequence of the fact that the distance of the centre of the n-fold $(n + 1)$-hedroid to one of its summits is $\frac{n}{n+1}$ times the distance from a summit to the centre of the opposite $(n - 1)$-fold boundary. The second expression is obtained from the equation $(2\,r_n)^2 = nl_n^2$, where l_n is the length of edge, and the third is the quadrant of a great circle on the n-fold sphere. Observe that in an infinite-dimensional space the second expression is zero and the first and third are iden-

Cases (*i*), (*ii*), (*iv*), give the binomial tri-group for five-dimensional space. Their Φ functions are respectively:

$$(1-1)^6 = 0, \quad 1-(2-1)^5 = 0, \quad (1-2)^5 + 1 = 0.$$

Cases (*v*), (*vii*), and (*viii*) each give rise to infinite figures, the first being self-reciprocal, the other two forming a reciprocal pair. The figure of Case (*v*) is infinite, for the edge of the frame figure is $\sqrt{2}$, hence that of the generating figure is equal to the radius of the circumscribed 5-fold unit sphere. In applying the criterion to Case (*vii*), we notice that the extremities of the eight edges diverging from a summit of an ikosatetrahedroid terminate at the summits of a cube, and the edges of that cube are edges of the ikosatetrahedroid. Making these eight summits coincide with those of one of the cubes of an oktahedroid inscribed to the 4-fold sphere, then the adjacent summit of the ikosatetrahedroid — since the edge of the oktahedroid is equal to the radius of the sphere — will lie at the centre of the sphere. Evidently then other ikosatetrahedroids may be added to the figure *ad infinitum*, in such a way as to saturate the four-dimensional space. The centres of these ikosatetrahedroids are the summits of an infinite series of successive hexadekahedroids, which also fill to saturation the infinite four-dimensional space. This is the figure of Case (*viii*). It may also be shown to be infinite by the application of the criterion (*B*).*

Cases (*iii*), (*ix*), (*vi*), (*x*), (*xi*). The edge of the frame figure of Case (*iii*) being 1.58113, the calculated length of edge of the generating figure is .97722, or less than the radius of the circumscribed sphere. Hence the figure is imaginary and also its reciprocal of Case (*ix*).† Moreover, since the edge of the frame figure of Case (*vi*) is shorter than the corresponding edge in Case (*iii*),

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tical. The edge of the (120)-hedroid inscribed in the unit 4-fold sphere must be less than the edge of a regular dodekahedron inscribed in the unit 3-fold sphere; for the dodekahedral boundaries to the (120)-hedroid are themselves inscribed in *small* spheres whose radii are less than the radius of the circumscribed 4-fold sphere. Hence the limit $l < 1$ for the (120)-hedroid is justified. The limit $l < 1$ for the (600)-hedroid is verified in the last footnote to this page.

* The fact here brought to light that four-dimensional space may be built up with either hexadekahedroids or ikosatetrahedroids suggests another method for calculating the length of an edge of the ikosatetrahedroid. Let A and B (Fig. 22) be the centres of two adjacent hexadekahedroids out of 24 which have been joined together about a point O, the centre of a 4-fold sphere which circumscribes the ikosatetrahedroid whose summits are A, B, etc. The radius of the sphere is the same as of that which circumscribes the hexadekahedroid whose centre is at B, so that $OB = BF = r$. Let C be the centre of a tetrahedral boundary common to the two adjacent hexadekahedroids. Then C bisects AB and CF is perpendicular to AB. Now HF is an edge of the hexadekahedroids and is equal to $r\sqrt{2}$; and $AB = l$ is an edge of the ikosatetrahedroid; and we have

$$CF = HF\sqrt{\tfrac{3}{8}} = r\sqrt{\tfrac{3}{4}}\,;\ CF^2 + CB^2 = FB^2, \text{ or } \tfrac{3}{4}r^2 + \tfrac{1}{4}l^2 = r^2\,;\ \therefore\ l = r.$$

† This verifies the assumption on page 12, that the length of edge of the regular 4-fold (600)-hedroid is less than the radius of the circumscribed 4-fold sphere. For if it were equal to or greater than the radius of the sphere, the application of Criterion (*B*) to Case (*ix*) would give either an infinite or real figure.

the reciprocal figures of Cases (*vi*) and (*x*) are also imaginary; for the same reason the self-reciprocal figure of Case (*xi*) is imaginary.

Case (*xii*). The generating figure is bounded by tetrahedra, the edge of which (in applying the criterion) must be equal to the edge of the frame figure; i. e. it is less than the radius of the sphere. Hence the figure is imaginary.

Thus, according to the criterion (B) here used, there are only three regular 5-fold (simple) figures. Now if this be so, then, according to the same criterion, there can be no more than three belonging to any space of more than five dimensions. Suppose, in fact, that we have a set of the tri-group of n-fold figures, and wish to build up $(n+1)$-fold regular figures. The three figures in question, the n-fold $(n+1)$-hedroid, $(2n)$-hedroid and (2^n)-hedroid, have respectively for the numbers of the summits belonging to their $(n-1)$-fold boundaries:

$$(a)\quad (n)_0^{n-1} = n; \qquad (b)\quad (n)_0^{n-1} = 2^{n-1}; \qquad (c)\quad (n)_0^{n-1} = n.$$

Now the equation of Criterion (C), page 8, gives

$$(n)_1^0 N_0 = (n)_0^1 N_1 = 2\,N_1,$$

where $(n)_1^0$ is the number of edges radiating from a given summit, and we have for the first case $(n)_1^0(n+1) = 2\,\frac{n(n+1)}{2}$, for the second $(n)_1^0\,2^n = 2\,n\,2^{n-1}$, and for the third $(n)_1^0\,.\,2\,n = 2\,.\,2\,n(n-1)$, which give the group of values

$$(a)'\quad (n)_1^0 = n; \qquad (b)'\quad (n)_1^0 = n; \qquad (c)'\quad (n)_1^0 = 2\,(n-1).$$

For those arrangements in the formation of a regular angle which are possible to all spaces we are to select, in obedience to the criterion (A), the pairs of values $(n)_0^{n-1}$, $(n)_1^0$, which are identical in the two groups, and these are

$$(a)\,(a)' \qquad (a)\,(b)' \qquad (c)\,(a)' \qquad (c)\,(b)'.$$

The fourth pair gives the arrangement for the infinite self-reciprocal figure, and the other three give rise to the tri-group of regular figures in $(n+1)$-dimensional space. The other five combinations, $(a)\,(c)'$, $(b)\,(a)'$, $(b)\,(b)'$, $(b)\,(c)'$, $(c)\,(c)'$, are possible only when $n = 2, 1, 1, 2, 2$ respectively. Hence, if n be greater than 2, the only regular figures that can be produced out of combinations of the three figures of the tri-group are their own analogues in the higher spaces.

It will be obvious to the reader of this paper, that the methods herein employed are extremely liable to errors which might materially modify the conclusions drawn, and I shall be surprised if none are found;

> "For hard, hard, hard is it only not to tumble,
> So fantastical is the dainty metre."

I wish, in conclusion, to make grateful acknowledgment to my coworkers at this University, and especially to Dr. Story, for valuable suggestions.

Plate I

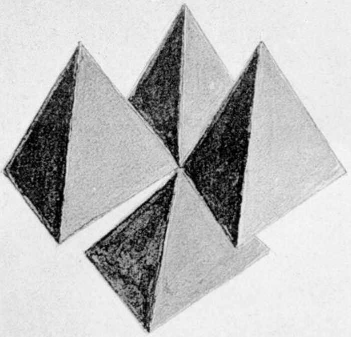

Fig. 1

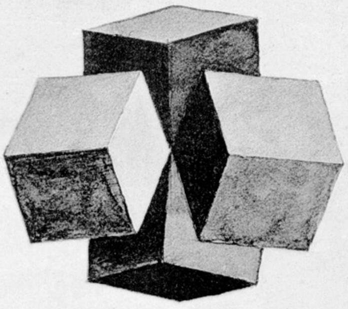

Fig. 3

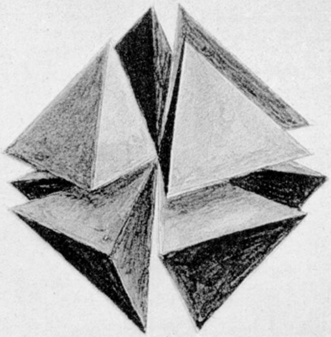

Fig. 5

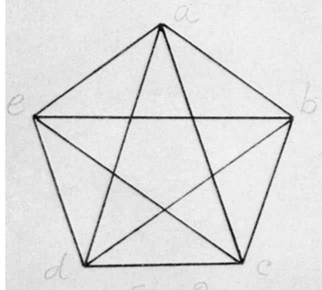

Fig. 2

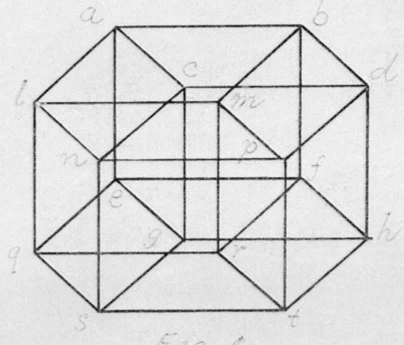

Fig. 4

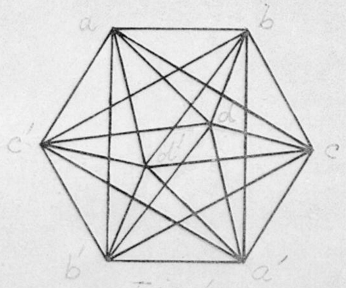

Fig. 6

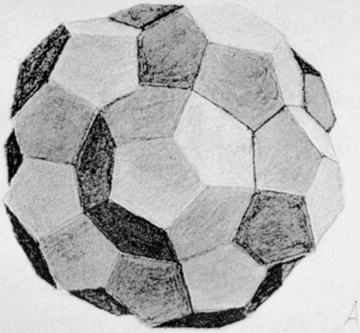

Fig. 18

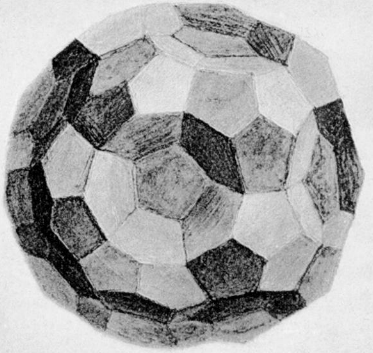

Fig. 19

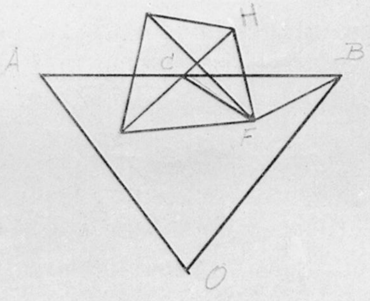

Fig. 22

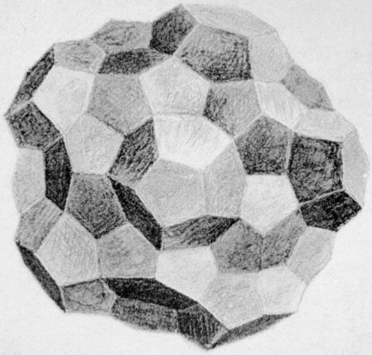

Fig. 20

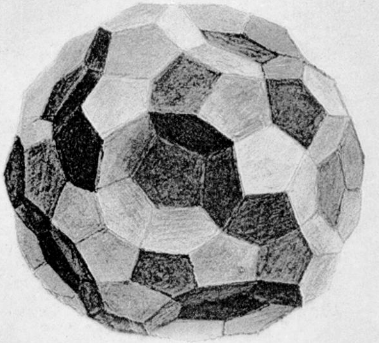

Fig. 21

Plate II.

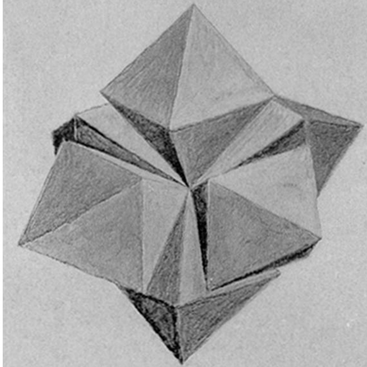

Fig. 7

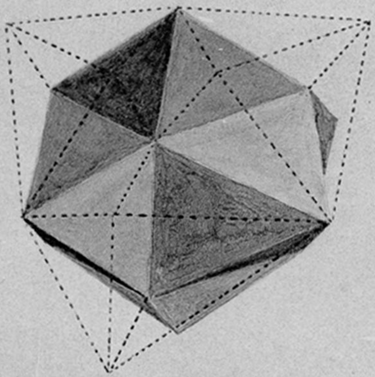

Fig. 8

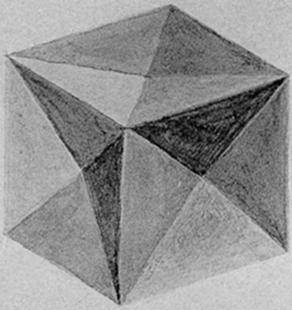

Fig. 9

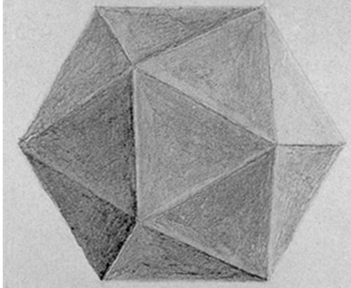

Fig. 10

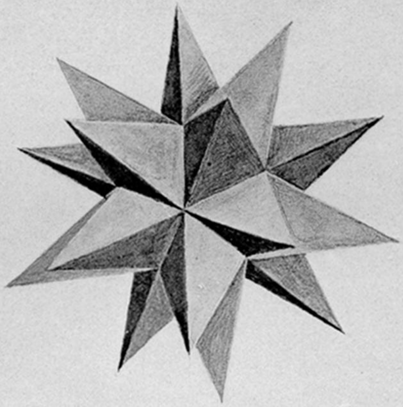

Fig. 11

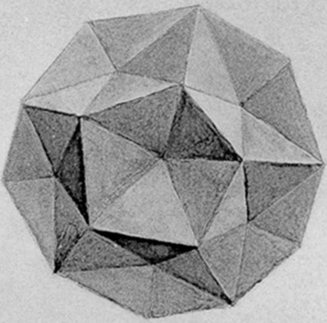

Fig. 12

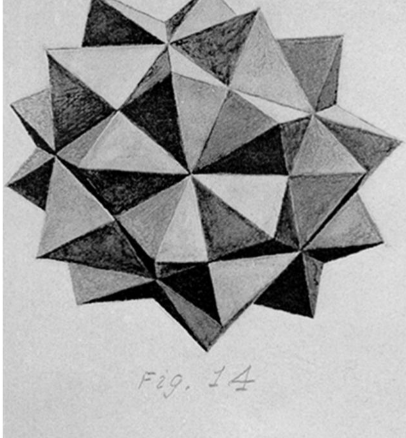

Fig. 14

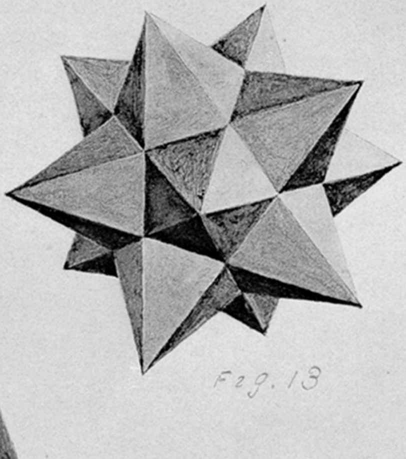

Fig. 13

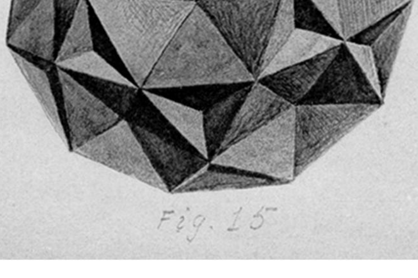

Fig. 15

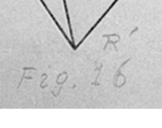

Fig. 16

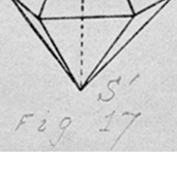

Fig 17